

## THE SPLITTABILITY AND TRIVIALITY OF 3-BRIDGE LINKS

BY

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**ABSTRACT.** A method to simplify 3-bridge projections of links and knots, called a *wave move*, is discussed in general situation and it is shown what kind of properties of 3-bridge links and knots can be recognized from their projections by wave moves. In particular, it will be proved that every 3-bridge projection of a splittable link or a trivial knot can be transformed into a disconnected one or a hexagon, respectively, by a finite sequence of wave moves. As its translation via the concept of 2-fold branched coverings of  $S^3$ , it follows that every genus 2 Heegaard diagram of  $S^2 \times S^2 \# L(p, q)$  or  $S^3$  can be transformed into one of specific standard forms by a finite sequence of operations also called wave moves.

**1. Introduction.** A *link* is a disjoint union of circles in a 3-sphere  $S^3$ . Especially a one-component link is called a *knot*. A link is said to be *splittable* if its link complement contains a 2-sphere which bounds no 3-ball, that is, if there is a 2-sphere which *splits* the link into two parts in  $S^3$ . A link is said to be *trivial* if it can be obtained as the boundary of a disjoint union of several disks embedded in  $S^3$ . There has been proposed by Haken [Ha] an algorithm to determine if a given link is splittable or trivial. His algorithm is based on the theory of normal surfaces and seeks for an incompressible 2-sphere or a boundary-compressing disk in the link complement with a fixed handle decomposition. It is, however, too difficult to carry it out practically. In this paper, we shall present a simple algorithm to recognize the splittability and triviality of a link described *on paper*, although we have to confine our objects to the 3-bridge links.

To explain it intuitively, we consider the link  $L$  as one in  $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ . Let  $\{b_1^+, \dots, b_n^+\}$  be two systems of arcs in the plane  $\mathbf{R}^2 = \mathbf{R}^2 \times \{0\}$  such that:

- (i)  $b_i^+ \cap b_j^+ = \emptyset$  and  $b_i^- \cap b_j^- = \emptyset$  ( $i \neq j$ ).
- (ii)  $\bigcup_{i=1}^n \partial b_i^+ = \bigcup_{i=1}^n \partial b_i^-$ .
- (iii) The interiors of  $b_i^+$  and  $b_j^-$  intersect transversely in at most finitely many points.

If a link  $L$  is equivalent to the one obtained from  $p(L) = \bigcup_{i=1}^n (b_i^+ \cup b_i^-)$  by pushing up  $b_i^+$  and down  $b_i^-$  slightly with their endpoints fixed, then  $p(L)$  is called an *n-bridge projection* of  $L$  on the plane  $\mathbf{R}^2$  with *over bridges*  $b_1^+, \dots, b_n^+$  and *under bridges*  $b_1^-, \dots, b_n^-$ . The projection  $p(L)$  may be considered as the image of  $L$  by the orthogonal projection of  $\mathbf{R}^3$  to  $\mathbf{R}^2$  after an ambient isotopic modification of  $L$ .

Every tame link admits an *n-bridge projection* for a finite number  $n$ . In particular, a trivial knot admits an *n-bridge projection* which has no crossing for all natural

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number  $n$ . Such an  $n$ -gonal projection is said to be *trivial*. On the other hand, a splittable link is represented by a *disconnected* projection on  $\mathbf{R}^2$  and a circle in  $\mathbf{R}^2$  which separates the projection is regarded as the intersection of  $\mathbf{R}^2$  with a 2-sphere which splits the link in  $\mathbf{R}^3$ . One will naturally expect an algorithm to transform a given projection into a trivial one for a trivial knot or into a disconnected one for a splittable link.

We shall define a strategy, called a *wave move*, to simplify an  $n$ -bridge projection  $p(L)$  of a link  $L$ . Let  $\omega$  be an arc in  $\mathbf{R}^2$  whose endpoints lie on a single bridge of  $p(L)$ , say an over bridge  $b_i^+$ , and whose interior does not meet  $p(L)$ , and let  $\beta$  be the subarc of  $b_i^+$  bounded by  $\partial\omega$ , which is often called the *subarc cut off from  $b_i^+$  by  $\omega$* . The replacement of  $b_i^+$  with  $(b_i^+ - \beta) \cup \omega$  yields a new  $n$ -bridge projection  $p'(L)$  of  $L$ . We call the arc  $\omega$  a *wave* and the transformation of  $p(L)$  into  $p'(L)$  a *wave move* if  $p'(L)$  has fewer crossings than  $p(L)$ , that is, if the interior of  $\beta$  contains at least one crossing of  $p(L)$ .

Homma and Ochiai [HO] have already shown the following result on the triviality of a knot:

**THEOREM 1-1 (HOMMA AND OCHIAI).** *Every 3-bridge projection of a trivial knot can be transformed into a trivial one by a finite sequence of wave moves.*

They proved the above as a consequence of a useful result on Heegaard splittings of genus 2 for  $S^3$ , stated as Theorem 1-3. On the other hand, we shall verify it, from a link-theoretical point of view, together with our original result on the splittability of a link:

**THEOREM 1-2.** *Every 3-bridge projection of a splittable link can be transformed into a disconnected one by a finite sequence of wave moves.*

Note that any disconnected 3-bridge projection can be modified, by succeeding wave moves, into a disjoint union of a trivial 1-bridge projection and one of Schubert's 2-bridge forms  $K(p, q)$  [ $\mathbf{S}_2$ ], where  $p$  and  $q$  are relatively prime integers and  $K(0, 1)$  represents a two-component trivial link.

These theorems suggest an algorithm for recognizing the triviality and splittability of a 3-bridge link; check whether the same bridge appears twice in the boundary of each region determined by a given projection, in order to seek for a wave, and repeat wave moves until a projection with no wave is obtained. This procedure will stop in a finite number of steps, since a wave move decreases strictly the number of crossings, and the final projection will have an expected form if the link is trivial or splittable.

The first two sections below contain the foundation of  $n$ -bridge presentation of a link in  $S^3$  for any positive integer  $n$ . We shall show in §2 that any two  $n$ -bridge projections which a fixed  $n$ -bridge decomposition of a link admits can be connected by a finite sequence of transformations called *jump moves* and, moreover, in §3 that every  $n$ -bridge decomposition of a trivial knot or a splittable link can be represented by a trivial or a disconnected  $n$ -bridge projection, respectively. The second fact might be misunderstood to be obvious but so it is not in our formulation of  $n$ -bridge

links, as is pointed out later. From these facts, it follows that every  $n$ -bridge projection of a trivial knot or a splittable link can be transformed into a promising projection by a finite sequence of jump moves. In §4 we shall change such a sequence of jump moves into that of wave moves, confining the *bridge number*  $n$  to 3. We shall prove axiomatically a more general proposition. Theorems 1-1 and 1-2 are only its immediate consequences.

As is well known, the 2-fold branched covering  $M^3$  of  $S^3$  branched over a 3-bridge link  $L$  has a Heegaard splitting of genus 2 (see [BH]). In particular, if  $L$  is trivial or splittable then  $M^3$  is homeomorphic to  $S^3$  or  $S^2 \times S^1 \# L(p, q)$ , where  $L(p, q)$  denotes a lens space of type  $(p, q)$ , including  $S^3$  and  $S^2 \times S^1$ . So our results can be translated, via this correspondence, into the following two theorems on Heegaard diagrams of genus 2, one of which Homma, Ochiai and Takahashi [HOT] have already proved:

**THEOREM 1-3 (HOMMA, OCHIAI AND TAKAHASHI).** *Every Heegaard diagram of genus 2 for  $S^3$  can be transformed into the standard one by a finite sequence of wave moves.*

**THEOREM 1-4.** *Every Heegaard diagram of genus 2 for  $S^2 \times S^1 \# L(p, q)$  can be transformed into one of the standard ones by a finite sequence of wave moves.*

The terminology for the above theorems are prepared in §5. Roughly speaking, a wave for a 3-bridge projection of a link is lifted to a wave for a genus 2 Heegaard diagram of its 2-fold branched covering.

In §6 we shall give some observations and examples concerning our results. They will suggest that it is hardly possible to recognize most well-known properties of links only by wave moves.

We work in the PL category. Quite standard notation and terminology are used without their definitions in this paper and can be found in [J] and [R].

**2.  $n$ -bridge links and jump moves.** In this section we shall give one formulation of  $n$ -bridge presentation of links and determine when two  $n$ -bridge projections represent a common link, roughly speaking.

Let  $L$  be a link in  $S^3$ . A 2-sphere  $S^2(L)$  transverse to  $L$  in  $S^3$  is called an  *$n$ -bridge decomposing sphere* of  $L$  if the following situation arises:

(i) The 2-sphere  $S^2(L)$  separates  $S^3$  into two 3-balls  $B^\pm$ , and  $B^\pm \cap L$  consist of  $n$  arcs  $b_1^\pm, \dots, b_n^\pm$ , respectively. Let  $S^3$  and  $S^2(L)$  be oriented; then the *upper half ball*  $B^+$  and the *lower half ball*  $B^-$  are distinguished uniquely. Each arc  $b_i^+$  and  $b_i^-$  is called an *over* and an *under bridge* of  $L$ , respectively.

(ii) There exist  $n$  pairwise disjoint disks  $D_1^+, \dots, D_n^+$  in  $B^+$ , called *over-bridge-spanning disks*, such that  $\partial D_i^+ \supset b_i^+$  and  $\text{cl}(\partial D_i^+ - b_i^+) = D_i^+ \cap S^2(L)$  ( $i = 1, \dots, n$ ) and there exist  $n$  similar disks  $D_1^-, \dots, D_n^-$  in  $B^-$  called *under-bridge-spanning disks*.

The composite structure  $(L, S^2(L); D_1^+, \dots, D_n^+; D_1^-, \dots, D_n^-)$  of  $L$  with decomposing sphere and bridge-spanning disks is an  *$n$ -bridge form* of  $L$ , and  $p(L) = (\cup_{i=1}^n (D_i^+ \cup D_i^-)) \cap S^2(L)$  is the  *$n$ -bridge projection* on  $S^2(L)$  of  $L$  associated with the  $n$ -bridge form. Each arc  $p(b_i^\pm) = D_i^\pm \cap S^2(L)$  on  $S^2(L)$  is called the *projection*

of an over or under bridge  $b_i^\pm$  and often an *over* or *under bridge* of the  $n$ -bridge projection  $p(L)$ . We shall assume that the over bridges and the under bridges of an  $n$ -bridge projection intersect *transversely* in their finitely many (or no) interior points and in their endpoints. That is, each crossing of a projection of a link is like the letter “X”.

A link  $L$  is said to be an  $n$ -bridge one if  $L$  admits an  $n$ -bridge decomposing sphere  $S^2(L)$ . We shall always take into account only one fixed  $n$ -bridge decomposing sphere  $S^2(L)$  when a link  $L$  is declared as an  $n$ -bridge link. Two  $n$ -bridge links  $L_1$  and  $L_2$  with decomposing spheres  $S^2(L_1)$  and  $S^2(L_2)$  are *equivalent* if there is an orientation-preserving homeomorphism  $h: S^3 \rightarrow S^3$  such that  $h(L_1) = L_2$ ,  $h(S^2(L_1)) = S^2(L_2)$  and  $h|_{S^2(L_1)}$  is also orientation-preserving. Notice that two  $n$ -bridge links with different decomposing spheres may not be equivalent as  $n$ -bridge links even if they have the same link type.

Two  $n$ -bridge forms of  $L_1$  and  $L_2$  are *equivalent* if there is an orientation-preserving homeomorphism  $h: S^3 \rightarrow S^3$  which makes the  $n$ -bridge link  $L_1$  equivalent to the  $n$ -bridge link  $L_2$  and which sends over- and under-bridge-spanning disks of  $L_1$  onto those of  $L_2$ , respectively. Two  $n$ -bridge projections  $p(L_1)$  on  $S^2(L_1)$  of  $L_1$ , and  $p(L_2)$  on  $S^2(L_2)$  of  $L_2$  are *equivalent* if there is an orientation-preserving homeomorphism  $h': S^2(L_1) \rightarrow S^2(L_2)$  which carries over and under bridges of  $p(L_1)$  onto those of  $p(L_2)$ , respectively. Such a homeomorphism  $h': S^2(L_1) \rightarrow S^2(L_2)$  extends to a homeomorphism  $h: S^3 \rightarrow S^3$  which gives the equivalence between  $n$ -bridge forms of  $L_1$  and  $L_2$ . Therefore there is a bijective correspondence between the equivalence classes of  $n$ -bridge forms and those of  $n$ -bridge projections of links.

In a usual way, a projection of a link, considered as one in  $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ , is obtained as its orthogonal projection on  $\mathbf{R}^2 \times \{0\}$  and ambient isotopic transformations of the link itself yield various projections. On the other hand, we consider that the variation of projections of a link depends on choice of its decomposing sphere and its bridge-spanning disks. Then we shall define an operation which corresponds to a change of bridge-spanning disks.

Let  $L$  be an  $n$ -bridge link with decomposing sphere  $S^2(L)$ , over bridges  $b_1^+, \dots, b_n^+$  and under bridges  $b_1^-, \dots, b_n^-$ , and let  $p(L)$  be an  $n$ -bridge projection on  $S^2(L)$  of  $L$  associated with an  $n$ -bridge form  $(L, S^2(L); D_1^+, \dots, D_n^+; D_1^-, \dots, D_n^-)$  of  $L$ . Choose an arc  $\beta$  on  $S^2(L)$  for an over or under bridge of  $L$ , say  $b_1^+$ , so that  $\beta \cap (p(b_1^+) \cup \dots \cup p(b_n^+)) = \partial p(b_1^+) = \partial \beta$  and the circle  $p(b_1^+) \cup \beta$  separates the other bridges  $p(b_2^+), \dots, p(b_n^+)$  into two nonempty groups in  $S^2(L)$ . Then we say that the over bridge  $p(b_1^+)$  of  $p(L)$  *jumps* to the arc  $\beta$ . Clearly the jump of  $p(b_1^+)$  to  $\beta$  corresponds to the replacement of  $D_1^+$  with a new bridge-spanning disk for  $b_1^+$  and  $p'(L) = \beta \cup p(b_2^+) \cup \dots \cup p(b_n^+) \cup p(b_1^-) \cup \dots \cup p(b_n^-)$  is the projection on  $S^2(L)$  of  $L$  associated with the new  $n$ -bridge form of  $L$ . Then we call  $p'(L)$  a projection obtained from  $p(L)$  by an *over jump move* (or an *under jump move* when an under bridge of  $p(L)$  jumps) and write  $p(L) \xrightarrow{\text{jump}} p'(L)$ .

An  $n$ -bridge projection  $p(L)$  on a decomposing sphere  $S^2(L)$  of a link  $L$  is *normalized* if there is no region in  $S^2(L)$ , obtained as a component of  $S^2(L) - p(L)$ , which is bounded by precisely two edges like the shaded part of Figure 2-1. Such an

inessential region is called a *cancelling region* of  $p(L)$  and the obvious way to eliminate a cancelling region is *normalization*.

**PROPOSITION 2-1.** *Any two normalized projections of an  $n$ -bridge link on its decomposing sphere are related to each other by a finite sequence of jump moves, up to equivalence.*

**PROOF.** Let  $p_1(L)$  and  $p_2(L)$  be two normalized projections of an  $n$ -bridge link  $L$  on its decomposing sphere  $S^2(L)$  and let  $\mathfrak{B}_j^\pm = p_j(b_1^\pm) \cup \cdots \cup p_j(b_n^\pm)$  ( $j = 1, 2$ ) be the union of over bridges and of under bridges of  $p_j(L)$ , respectively. We shall first carry  $\mathfrak{B}_1^+$  onto  $\mathfrak{B}_2^+$  by over jump moves, neglecting  $\mathfrak{B}_2^-$ .

Define  $\alpha^\pm = \alpha^\pm(p_1(L))$  as the number of points in the intersection of  $\mathfrak{B}_1^\pm$  with  $\mathfrak{B}_2^+$  except their endpoints and assume that  $p_1(L)$  is reselected out of the equivalence class of  $p_1(L)$  so as to minimize the pair  $(\alpha^+, \alpha^-)$  with respect to the lexicographical order. Then  $\mathfrak{B}_1^+ \cup \mathfrak{B}_2^+$  and  $\mathfrak{B}_1^- \cup \mathfrak{B}_2^-$  are normalized, formally considered as projections on  $S^2(L)$  of certain  $n$ -bridge links.

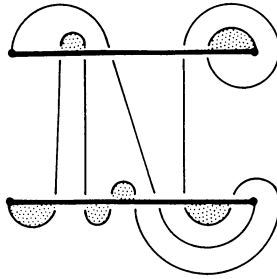


FIGURE 2-1

If  $\alpha^+ > 0$  then we can find an over bridge  $p_1(b_i^+)$  of  $p_1(L)$  and a subarc  $\gamma$  of an over bridge  $p_2(b_j^+)$  of  $p_2(L)$  such that  $\mathfrak{B}_1^+ \cap \gamma = \partial\gamma \subset p_1(b_i^+)$  and at least one point of  $\partial\gamma$  lies in the interior of  $p_1(b_i^+)$ , considering the intersection of over-bridge-spanning disks of two  $n$ -bridge forms associated with  $p_1(L)$  and  $p_2(L)$ . Let  $\gamma'$  be the subarc of  $p_1(b_i^+)$  cut off by  $\partial\gamma$  then  $\gamma \cup \gamma'$  must be a circle which separates the other over bridges of  $p_1(L)$  in  $S^2(L)$ , because of the minimality of  $p_1(L)$ . Thus  $p_1(b_i^+)$  can jump to an arc along  $(p_1(b_i^+) - \gamma') \cup \gamma$  so that the projection of  $L$  obtained from  $p_1(L)$  by the jump move is normalized and  $\alpha^+$  decreases. If  $\alpha^+ > 0$  again, repeat the above process until  $\alpha^+ = 0$ . In the final case, it is easy to find a sequence of jump moves which transforms  $p_1(L)$  into an  $n$ -bridge projection on  $S^2(L)$  of  $L$  equivalent to  $\mathfrak{B}_2^+ \cup \mathfrak{B}_1^-$ .

Now the over bridges of  $p_1(L)$  has coincided with those of  $p_2(L)$ . By the similar argument, the under bridges of  $p_1(L)$  will be carried onto those of  $p_2(L)$  by a sequence of under jump moves, up to equivalence. ■

The above proof shows that each jump move of the sequence in the proposition can be chosen not to break the normality of projections. Notice that there is no jump move for 1- and 2-bridge normalized projections of links, which implies that

**COROLLARY 2-2.** *Every 1- or 2-bridge link has a unique normalized projection on its decomposing sphere, up to equivalence.*

In case of  $n \geq 3$ , the hypothesis that two  $n$ -bridge projections are normalized can be omitted from Proposition 2-1. Because a single normalization can be replaced with two jump moves; first jump one of the bridges which hold a cancelling region to any position, and next jump it back to the position of it after the normalization.

**COROLLARY 2-3.** *Any two projections of an  $n$ -bridge link on its decomposing sphere are joined to each other by a finite sequence of jump moves, up to equivalence, if  $n \geq 3$ .*

**REMARK.** Proposition 2-1 does not assert that any two  $n$ -bridge projections of a link are joined to each other by a finite sequence of jump moves. Projections of two inequivalent  $n$ -bridge links cannot be related by jump moves at all even if they represent the same link.

Let  $L$  be an  $n$ -bridge link with decomposing sphere  $S^2(L)$  and  $L'$  the  $n$ -bridge link whose link type is the same one as  $L$  and whose decomposing sphere  $S^2(L')$  is  $S^2(L)$  with opposite orientation. Suppose that  $L$  and  $L'$  are inequivalent as  $n$ -bridge links, then they do not have equivalent  $n$ -bridge projections unless the role of over and under bridges are interchanged. For example, we can take the 2-bridge knot  $K(7, 2)$  as  $L$ ; then  $L'$  is equivalent to the 2-bridge knot  $K(7, 4)$ . By the uniqueness of normalized projections of 2-bridge knots (Corollary 2-2),  $K(7, 2)$  and  $K(7, 4)$  are not equivalent as 2-bridge knots in our sense.

Now consider the *splittable union*  $L \cup L$  and  $L \cup L'$  whose decomposing spheres are obtained naturally as the oriented connected sums  $(S^3, S^2(L)) \# (S^3, S^2(L))$  and  $(S^3, S^2(L)) \# (S^3, S^2(L'))$ , respectively. It is easy to see that these links  $L \cup L$  and  $L \cup L'$  have the same link type but that they are inequivalent as  $2n$ -bridge links. (Use Proposition 3-2 in the next section.) Moreover, none of  $2n$ -bridge projections of  $L \cup L$  can be transformed into a  $2n$ -bridge projection of  $L \cup L'$  by jump moves even if the interchange between over and under bridges is allowed, because they are inequivalent as  $2n$ -bridge links even if orientation-reversing equivalence of the decomposing spheres is allowed.

Note that the same example can be given from  $L \# L$  and  $L \# L'$  which are  $(2n - 1)$ -bridge links. (Use Proposition 3-5.)

There have been given more essential examples of the above remark by Montesinos [Mon]. In his paper,  $n$ -bridge decompositions are called *2n-plat presentations*, and inequivalent 3-bridge decompositions which exhibit equivalent *prime* knots or links of bridge index 3 are constructed.

**3.  $n$ -bridge forms and 2-spheres.** As is pointed out in the previous section, two  $n$ -bridge projections of a link may not be joined by jump moves if they are projected to different  $n$ -bridge decomposing spheres. So one may ask whether a given decomposing sphere admits a trivial projection for a trivial knot, or a disconnected projection for a splittable link.

Otal [O] has already answered this question for a trivial knot. He showed that any two trivial knots with  $n$ -bridge decomposing spheres are equivalent as  $n$ -bridge knots, using a technique of Morse functions.

In other words, his result states that

**PROPOSITION 3-1 (OTAL).** *Given an  $n$ -bridge decomposing sphere  $S^2(K)$  of a trivial knot  $K$ , then there is an  $n$ -bridge form of  $K$  with decomposing sphere  $S^2(K)$  which gives a trivial projection on  $S^2(K)$ .*

In this section, we shall give the affirmative answer to the question in case of a splittable link.

**PROPOSITION 3-2.** *Given an  $n$ -bridge decomposing sphere  $S^2(L)$  of a splittable link  $L$ , then there is an  $n$ -bridge form of  $L$  with decomposing sphere  $S^2(L)$  which gives a disconnected projection on  $S^2(L)$ .*

To prove this, it should be observed how a 2-sphere which splits  $L$  intersects an  $n$ -bridge decomposing sphere. In a more general situation, we shall establish the following lemma, using a method of three-manifold topology:

**LEMMA 3-3.** *Let  $L$  be an  $n$ -bridge link with decomposing sphere  $S^2(L)$  and  $S_0^2$  a 2-sphere in  $S^3$  which meets  $L$  transversely in  $m$  points. (Necessarily  $m$  must be even.) Suppose that  $S_0^2 - \dot{U}(L)$  is incompressible and boundary-incompressible in  $S^3 - \dot{U}(L)$ , where  $\dot{U}(L)$  denotes an open tubular neighborhood of  $L$  in  $S^3$ . Then there exists a 2-sphere  $S^2$  in  $S^3$  and an  $n$ -bridge form  $(L, S^2(L); D_1^+, \dots, D_n^+; D_1^-, \dots, D_n^-)$  of  $L$  such that:*

- (i)  $S^2$  meets  $L$  transversely in  $m$  points.
- (ii)  $S^2 - \dot{U}(L)$  is incompressible and boundary-incompressible in  $S^3 - \dot{U}(L)$ .
- (iii)  $S^2$  meets  $S^2(L)$  transversely along a single circle.
- (iv)  $S^2 \cap D_i^+$  ( $i = 1, \dots, n$ ), if nonempty, consists of arcs each of which joins a point of  $S^2 \cap L$  to a point on  $S^2(L)$ . If  $m = 0$  or 2 then  $S^2 \cap D_i^-$  ( $i = 1, \dots, n$ ) also consists of such arcs.

*When  $L$  is not splittable then  $S^2$  can be chosen to be sent to  $S_0^2$  by an ambient isotopy of  $(S^3, L)$ .*

**PROOF.** We shall give an algorithmical proof. Let  $S^2$  be a 2-sphere in  $S^3$  satisfying conditions (i) and (ii), say  $S_0^2$ , and let  $(L, S^2(L); D_1^+, \dots, D_n^+; D_1^-, \dots, D_n^-)$  be an  $n$ -bridge form of  $L$  with decomposing sphere  $S^2(L)$ . By the general position argument, it may be always assumed that  $S^2 \cap S^2(L) \cap L = \emptyset$  and that  $S^2$  intersects  $S^2(L)$  transversely along a finite number of circles. The following procedure will transform the 2-sphere  $S^2$  and the  $n$ -bridge form of  $L$  into the desired ones.

Set  $T^\pm = S^2 \cap B^\pm$ , where  $B^\pm$  are two balls into which  $S^2(L)$  splits  $S^3$ . If  $T^+ - \dot{U}(L)$  has a compressing disk  $D^2$  in  $B^+ - \dot{U}(L)$  then  $\partial D^2$  bounds a disk  $E^2$  in  $S^2 - \dot{U}(L)$  since  $S^2 - \dot{U}(L)$  is incompressible in  $S^3 - \dot{U}(L)$ . The disk  $E^2$  must intersect  $S^2(L)$  in several circles. Replace  $E^2$  with  $D^2$  in  $S^2$ ; then the number of components of  $S^2 \cap S^2(L)$  decreases. It is easy to see that the new  $S^2$  satisfies conditions (i) and (ii). (A suitable one should be chosen as  $E^2$  out of two disks

bounded by  $\partial D^2$  in  $S^2$  if  $m = 0$ .) Do the similar procedure when  $T^- - \dot{U}(L)$  is compressible in  $B^- - \dot{U}(L)$ . If  $L$  is not splittable, that is, if the link complement  $S^3 - \dot{U}(L)$  is irreducible, then the interchange between two disks which form a 2-sphere can be realized by an ambient isotopy of  $(S^3, L)$ .

Hereafter, the planar surfaces  $T^\pm - \dot{U}(L)$  are assumed to be incompressible in  $B^\pm - \dot{U}(L)$  and they will be modified by only ambient isotopies of  $(S^3, L)$ .

Suppose that one of  $T^\pm$ , say  $T^+$ , is not a disjoint union of several disks, then so  $T^+ - \dot{U}(L)$  is not. Notice that the disks  $D_i^+ - \dot{U}(L)$  ( $i = 1, \dots, n$ ) are meridian disks of the handlebody  $B^+ - \dot{U}(L)$ . Since  $T^+ - \dot{U}(L)$  is incompressible in  $B^+ - \dot{U}(L)$ ,  $T^+$  must intersect essentially at least one of the bridge-spanning disks  $D_1^+, \dots, D_n^+$ . By the general position argument again, it may be assumed that  $T^+ \cap D_i^+$  consists of two types of arcs; one is an arc, called type 1, which joins a point of  $T^+ \cap L$  to a point on  $S^2(L) \cap D_i^+$ , and the other, called type 2, joins two points on  $S^2(L) \cap D_i^+$ . A loop in the interior of  $D_i^+$ , if it exists, is immediately removable from  $T^+ \cap D_i^+$  by the modification of  $D_i^+$ 's because of the incompressibility of  $T^+ - \dot{U}(L)$  in  $B^+ - \dot{U}(L)$ . No arc joining two points of  $T^+ \cap L$  exists in  $T^+ \cap D_i^+$  since  $S^2 - \dot{U}(L)$  is boundary-incompressible in  $S^3 - \dot{U}(L)$ . An intersecting arc running in the disk  $D_i^+$  is said to be *innermost* if it cuts off a piece of  $D_i^+$  which is adjacent to  $S^2(L) \cap D_i^+$  and which contains no other intersecting arc, as usual.

Now we use the following three modifications, illustrated in Figures 3-1, 3-2 and 3-3: Modification 1<sup>+</sup> is a transformation of a neighborhood of an innermost intersecting arc  $\gamma$  of type 1. It pushes up (or down) the arc  $\gamma$  isotopically into  $B^-$  to remove a point of  $T^+ \cap L$  out of  $T^+$ . The homeomorphic type of  $T^+$  is unchanged while  $T^+ - \dot{U}(L)$  is cut by an essential arc. In general, we call a spanning arc of a surface an *inessential* one if it cuts off a disk from the surface and otherwise an *essential* one. Modification 2<sup>+</sup> is applied to a neighborhood of an innermost intersecting arc  $\gamma$  of type 2 only when  $\gamma$  is an essential arc of  $T^+$  and hence of  $T^+ - \dot{U}(L)$ . This modification is called an *isotopy of type A* by Jaco in his book [J] and is corresponding to the cutting of  $T^+$  along an essential arc in a *hierarchy* for the planar surface  $T^+$  (see, [J, Chapter II]). Modification 3<sup>+</sup> does not transform  $T^+$  but  $D_i^+$  to eliminate an intersecting arc  $\gamma$  of type 2 which is inessential in  $T^+$ . The arc  $\gamma$

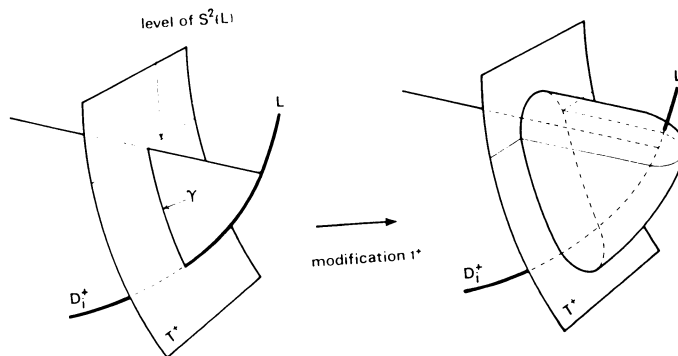


FIGURE 3-1



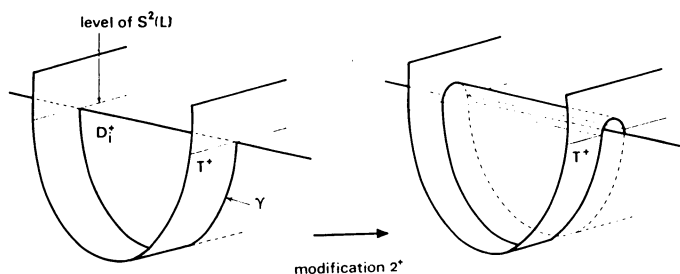


FIGURE 3-2

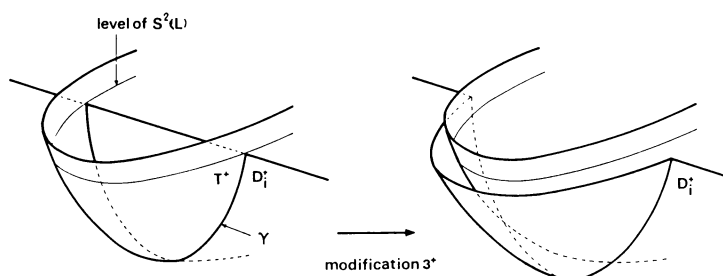


FIGURE 3-3

is required to cut off a disk from  $T^+ - \dot{U}(L)$  which contains no other intersecting arc but  $\gamma$  may not be innermost in  $D_i^+$ . Do these modifications repeatedly as possible.

Suppose that modifications  $1^+$  and  $2^+$  are not applicable; then all innermost intersecting arcs in each  $D_i^+$  are inessential in  $T^+$ . Let  $\gamma'$  be one such arc. If  $\gamma'$  is inessential also in  $T^+ - \dot{U}(L)$  then the situation where modification  $3^+$  is applicable will be easily found. Now assume that  $\gamma'$  is essential in  $T^+ - \dot{U}(L)$ , that is, the disk  $\Delta^2$  cut off by  $\gamma'$  from  $T^+$  contains at least one point of  $T^+ \cap L$  and an intersecting arc  $\gamma_1$  of type 1 starting from the point. Let  $D^2$  be the innermost disk in  $D_i^+$  cut off by  $\gamma'$ ; then  $D^2 \cup \Delta^2$  forms a disk which splits the ball  $B^+$  into two balls. We consider each of these two balls as the *outside* or the *inside* of  $D^2 \cup \Delta^2$  according to whether the ball does or does not contain the over bridge  $D_i^+ \cap L$ . Since the arc  $\gamma_1$  of type 1 is not innermost in the bridge-spanning disk  $D_j^+$  which includes  $\gamma_1$ , the inside ball must contain an inessential intersecting arc  $\gamma''$  of type 2 lying in  $D_j^+$  and the disk cut off by  $\gamma''$  from  $T^+$ . Then we consider that  $\gamma''$  is more inner than  $\gamma'$ . Doing the innermost argument in this sense, we can find an inessential intersecting arc  $\gamma$  of type 2 which has a neighborhood described in Figure 3-3. Therefore, one of modifications  $1^+$ ,  $2^+$  and  $3^+$  is applicable as far as  $T^+$  meets  $D_1^+ \cup \dots \cup D_n^+$ .

It is easy to see that modifications  $1^+$ ,  $2^+$  and  $3^+$  do not break the incompressibility of  $T^+ - \dot{U}(L)$  in  $B^+ - \dot{U}(L)$ , and hence the repetition of these modifications will be continued until  $T^+$  and  $T^+ - \dot{U}(L)$  are finally transformed into a disjoint union of several disks which are contained in the ball  $B^+ - \dot{U}(D_1^+ \cup \dots \cup D_n^+)$ . This

implies that the consecutive modifications realize a hierarchy for the planar surface  $T^+$ . By Lemma II.8 in [J], the number of the final disks is less than the number of boundary components of the original  $T^+$ . Since  $\partial T^+ = S^2 \cap S^2(L)$ , the number of circles of  $S^2 \cap S^2(L)$  decreases after the repetition of modifications  $1^+$ ,  $2^+$  and  $3^+$ .

Now each component of  $T^+$  has been a disk, while  $T^-$  may have a component not being a disk. Then return to the first step and repeat the above process, alternatively for  $T^+$  and  $T^-$ , until both of  $T^\pm$  become disjoint unions of disks. Since the process decreases the number of circles of  $S^2 \cap S^2(L)$  strictly, its repetition will stop in a finite number of steps and  $S^2 \cap S^2(L)$  will be a single circle. So we got a 2-sphere  $S^2$  and an  $n$ -bridge form of  $L$  with decomposing sphere  $S^2(L)$  which satisfy conditions (i), (ii) and (iii).

To make them satisfy the fourth condition (iv), first apply modifications  $1^+$  and  $3^+$  at random, then  $S^2 \cap (D_1^+ \cup \cdots \cup D_n^+)$  will be empty. Modifications  $1^+$  and  $3^+$  do not change the number of components of  $S^2 \cap S^2(L)$ , which is equal to 1 now, while modification  $2^+$  is not applicable since the disk  $T^+$  contains no essential spanning arc. In case of  $m > 2$ , we have the conclusion. When  $m = 0$ , eliminate the intersection of  $S^2$  with the under-bridge-spanning disks  $D_1^-, \dots, D_n^-$  by only modification  $3^-$ , which is  $3^+$  for  $D_i^-$ , not changing the 2-sphere  $S^2$ . When  $m = 2$ , first apply modifications  $1^-$  and  $3^-$  repeatedly until the two points of  $S^2 \cap L$  lie in the two disks  $T^\pm$  separately. Since intersecting arcs of type 2 in  $T^\pm$  cut off at least two outermost disks from  $T^\pm$ , respectively, modification  $3^\pm$  can be continued until

$$S^2 \cap \left( \bigcup_{i=1}^n (D_i^+ \cup D_i^-) \right)$$

consists of only two arcs of type 1. Now the desired ones have been obtained. ■

Proposition 3-2 is nothing but Lemma 3-3 in case of  $m = 0$ , where we take as  $S_0^2$  a 2-sphere which splits the link  $L$ . Combining it with Otal's result, we have

**COROLLARY 3-4.** *Given an  $n$ -bridge decomposing sphere  $S^2(L)$  of a trivial link  $L$ , then there is an  $n$ -bridge form of  $L$  with decomposing spheres  $S^2(L)$  which gives a projection each of whose component is a trivial one.*

In case of  $m = 2$ , Lemma 3-3 gives us information about the relationship between bridge decompositions and connected sum decompositions of a link. Let  $B(L_i)$  be an  $n_i$ -bridge form of a link  $L_i$  with decomposing sphere  $S^2(L_i)$  ( $i = 1, 2$ ) and let  $D_{i1}^\pm$  be its over- and under-bridge-spanning disks which are adjoining in a point  $x_i$  of  $S^2(L_i) \cap L_i$ . Choose a sufficiently small ball  $B_i^3$  in  $S^3$  containing the point  $x_i$  as its center so that  $B_i^3 \cap S^2(L_i)$  is a disk which contains no crossing of the projection associated with the  $n_i$ -bridge form  $B(L_i)$  and that the intersection of  $B_i^3$  with the bridge-spanning disks of  $B(L_i)$  consists of only two disks  $B_i^3 \cap D_{i1}^\pm$  which meet in a single point  $x_i$ . Remove the interior of the balls  $B_1^3$  and  $B_2^3$  from two copies of  $S^3$ , respectively, and identify the resulting 2-sphere boundaries  $\partial B_1^3$  and  $\partial B_2^3$  so that the arcs  $\partial B_1^3 \cap D_{11}^\pm$  coincides with  $\partial B_2^3 \cap D_{21}^\pm$ , respectively. Then an  $(n_1 + n_2 - 1)$ -bridge form of the connected sum  $L_1 \# L_2$  in the 3-sphere  $S^3 \# S^3$  will be obtained. We call it a *connected sum* of two bridge forms  $B(L_1)$  and  $B(L_2)$ . Lemma 3-3 with

$m = 2$  can be translated into the following

**PROPOSITION 3-5.** *Given an  $n$ -bridge decomposing sphere  $S^2(L)$  of a composite link  $L = L_1 \# L_2$ , then there is an  $n$ -bridge form of  $L$  with decomposing sphere  $S^2(L)$  which decomposes into a connected sum of bridge forms of the link  $L_1$  and  $L_2$ .*

The *bridge index*  $b(L)$  of a link  $L$  is defined as the minimum number  $n$  such that  $L$  admits an  $n$ -bridge decomposing sphere. As immediate consequences of Propositions 3-2 and 3-5, we have the following well-known formulas about the additivity of bridge indices of links:

**COROLLARY 3-6.** (i)  $b(L_1 \cup L_2) = b(L_1) + b(L_2)$ .  
(ii) (*H. Schubert* [ $S_1$ ])  $b(L_1 \# L_2) = b(L_1) + b(L_2) - 1$ .

**4. Wave-admissible properties.** In the previous two sections, we have observed that every  $n$ -bridge projection of a trivial knot and of a splittable link is joined to a trivial one and a splittable one, respectively, by a finite sequence of jump moves. Our purpose in this section is to translate such a sequence of jump moves into that of wave moves and to give a proof of Theorems 1-1 and 1-2. The triviality and the splittability of 3-bridge links will be treated together axiomatically as *wave-admissible properties*.

For a technical reason, we classify waves into two types, called  $p$ -waves and  $s$ -waves. Let  $p(L)$  be a projection of an  $n$ -bridge link  $L$  on its decomposing sphere  $S^2(L)$  with over bridges  $p(b_i^+)$  and under bridges  $p(b_i^-)$  ( $i = 1, \dots, n$ ). A  $p$ -wave along an over bridge  $p(b_i^+)$  is an arc  $\omega$  in  $S^2(L)$ , as is illustrated in Figure 4-1, joining an under bridge  $p(b_j^-)$  which holds a cancelling region together with the over bridge  $p(b_i^+)$ . The replacement of the subarc of  $p(b_j^-)$  around the cancelling region with the  $p$ -wave  $\omega$  is called a  $p$ -wave move, and it is nothing but normalization. If  $p(L)$  has a  $p$ -wave along an over bridge then  $p(L)$  has also a  $p$ -wave along an under bridge and those two  $p$ -wave moves have the same effect on  $p(L)$ . An  $s$ -wave joining an over bridge  $p(b_i^+)$  is an arc  $\omega$  in  $S^2(L)$  such that:

(i) The endpoints of  $\omega$  lie in the interior of  $p(b_i^+)$  and  $\omega \cap p(L) = \omega \cap p(b_i^+) = \partial\omega$ .

(ii) The subarc  $\beta$  of  $p(b_i^+)$  bounded by  $\partial\omega$  contains at least one crossing in its interior.

(iii) If the interior of  $\beta$  contains precisely two crossings then the circle  $\beta \cup \omega$  separates the other over bridges, except  $p(b_i^+)$ , in  $S^2(L)$ .

That is, an  $s$ -wave is a wave different from a  $p$ -wave. An  $s$ -wave move with  $\omega$  replaces the bridge  $p(b_i^+)$  with the new bridge  $(p(b_i^+) - \beta) \cup \omega$ .

When a projection  $p'(L)$  is obtained from a projection  $p(L)$  by a wave move, we shall write  $p(L) \xrightarrow{\omega} p'(L)$ , indicating the used wave  $\omega$ . In particular, if the wave  $\omega$  is a  $p$ -wave then the wave move will be written simply by  $p(L) \xrightarrow{p} p'(L)$ . A sequence of wave moves

$$p_1(L) \xrightarrow{\omega_1} p_2(L) \xrightarrow{\omega_2} \cdots \xrightarrow{\omega_n} p_{n+1}(L)$$

is said to be *regular* provided that if  $\omega_i$  is an  $s$ -wave then  $p_i(L)$  has no  $p$ -wave for all  $i = 1, \dots, n$ . In other words, a regular sequence of wave moves, often called simply

*regular wave moves*, is a sequence obtained when we apply wave moves to projections, making it a rule to normalize them primarily as possible and to use  $s$ -waves secondly for normalized ones.

A property  $Q$  of projections of an  $n$ -bridge link on its decomposing sphere is *W-admissible* (*wave-admissible*) if the following two conditions are satisfied:

(i) If a projection  $p(L)$  can be transformed into a projection  $p'(L)$  which has the property  $Q$  by a single jump move, then there is a regular sequence of wave moves which transforms  $p(L)$  into a projection  $p''(L)$  which also has the property  $Q$ . (Possibly  $p'(L)$  and  $p''(L)$  are not equivalent.)

(ii) If a projection  $p(L)$  has the property  $Q$  then any equivalent projection or any projection obtained from  $p(L)$  by a  $p$ -wave move has the property  $Q$ .

The first condition (i) is a necessary condition for a sequence of jump moves to be translated into a sequence of wave moves and it will assume the first step of the inductive proof of our main lemma.

**PROPOSITION 4-1.** *The disconnectivity and the triviality of projections of an  $n$ -bridge link are W-admissible.*

**PROOF.** These properties clearly satisfy (ii), so we shall check only (i) for them.

*Disconnectivity.* Assume that a connected projection  $p(L)$  is deformed into a disconnected one  $p'(L)$  by a jump of an over bridge  $p(b_i^+)$ . There is a circle in  $S^2(L)$  which separates  $p'(L)$  and the over bridge  $p(b_i^+)$  intersects it transversely in an even number of points. Then consecutive waves can be found along the circle until their wave moves carry  $p(L)$  to a disconnected one, which may be inequivalent to  $p'(L)$ .

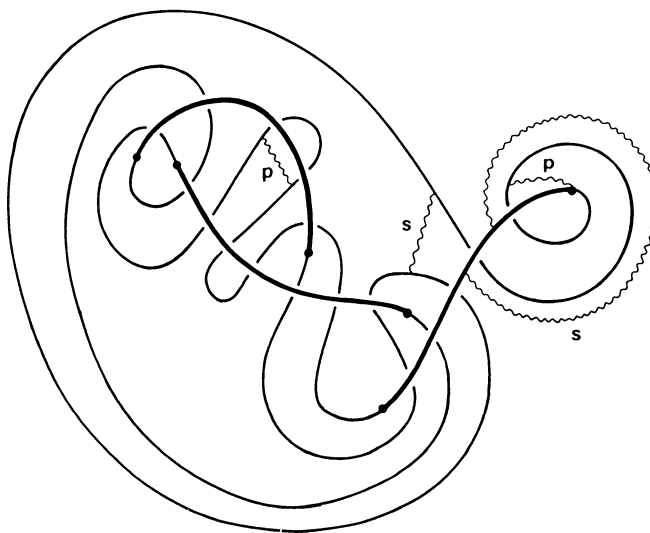


FIGURE 4-1

*Triviality.* If a nontrivial projection  $p(L)$  is transformed into a trivial one  $p'(L)$  by a jump of an over bridge  $p(b_i^+)$ , then  $p'(b_i^+)$  is regarded as a wave for  $p(L)$  and  $p'(L)$  is the result of the wave move. ■

Now the terminology to state the following main result has been prepared:

**PROPOSITION 4-2.** *Let  $L$  be a 3-bridge link with decomposing sphere  $S^2(L)$  and  $Q$  a  $W$ -admissible property of projections. If  $L$  admits a 3-bridge projection on  $S^2(L)$  which has the property  $Q$  then every 3-bridge projection on  $S^2(L)$  of  $L$  can be transformed into either a 3-bridge projection which has the property  $Q$  or a disconnected one by a regular finite sequence of wave moves.*

This proposition implies Theorems 1-1 and 1-2. We can take as the  $W$ -admissible property  $Q$  the triviality for Theorem 1-1 and the disconnectivity for Theorem 1-2. Proposition 3-1 or 3-2 assures the existence of a 3-bridge projection which has the property  $Q$  in each case.

To prove Proposition 4-2, it is sufficient to establish the next lemma.

**LEMMA 4-3.** *Let  $L$  be a 3-bridge link and  $Q$  a  $W$ -admissible property of projections. Given a jump move followed by a regular sequence of wave moves for 3-bridge projections of  $L$*

$$p_0(L) \xrightarrow{\text{jump}} p_1(L) \xrightarrow{\omega_1} p_2(L) \xrightarrow{\omega_2} \cdots \xrightarrow{\omega_n} p_{n+1}(L) \quad (n \geq 0)$$

*with  $p_{n+1}(L)$  having the property  $Q$ , then there is a regular finite sequence of wave moves which transforms the initial projection  $p_0(L)$  into either a 3-bridge projection which has the property  $Q$  or a disconnected one.*

By Corollary 2-3 and condition (ii), a finite sequence of jump moves joins a given 3-bridge projection  $p(L)$  of  $L$  to a 3-bridge projection with the property  $Q$  whose existence is assumed in Proposition 4-2. Using Lemma 4-3 repeatedly, we can change the jump moves into regular wave moves which carry  $p(L)$  to either a projection with the property  $Q$  or a disconnected one. Remark that the length of the sequence in the conclusion of Lemma 4-3, which is finite, cannot be estimated from the length  $n$  of the original sequence in general.

Now we shall give the main proof of our results.

**PROOF OF LEMMA 4-3.** We shall use induction on the complexity of a given sequence defined as the pair  $(|p_0(L)|, n)$  in the lexicographical order, where  $|p(L)|$  denotes the number of crossings in a projection  $p(L)$ . In case of  $n = 0$ , the lemma immediately holds, not depending on  $|p_0(L)|$ , by the definition of a  $W$ -admissible property. So the first step of our induction is assured necessarily.

Suppose the inductive hypothesis with  $n > 0$ . It may be assumed that  $p_0(L)$  is connected and  $p_{n+1}(L)$  is normalized. For simplifying the usage of symbols, we shall denote the three over bridges of  $p_0(L)$  by  $b_1, b_2$  and  $b_3$ , instead of  $p_0(b_1^+)$ ,  $p_0(b_2^+)$  and  $p_0(b_3^+)$ , and assume that the over bridge  $b_1$  jumps to an over bridge  $b'_1$  of  $p_1(L)$ . Notice that  $b_1$  and  $b'_1$  form a circle in  $S^2(L)$  which separates  $b_2$  from  $b_3$ , and that  $p_0(L)$  and  $p_1(L)$  have common bridges except  $b_1$  and  $b'_1$ . When we illustrate the situation by figures, the over bridges  $b_1$  and  $b'_1$  will be always drawn as a horizontal

segment and a lower half-circle joining two ends of the segment, respectively. We shall investigate the relationship of the initial jump move and the first wave  $\omega_1$  in the following various cases.

*Case 1.* Either  $p_0(L)$  or  $p_1(L)$  has a  $p$ -wave. We may assume that each  $p$ -wave lies along an over bridge of  $p_0(L)$  or  $p_1(L)$ . Choose an “innermost”  $p$ -wave  $\omega_0$  for the union of  $p_0(L)$  and  $p_1(L)$ . Then the  $p$ -wave  $\omega_0$  cuts off an arc from an under bridge so that the interior of the arc intersects only one of over bridges  $b_1, b_2, b_3$  and  $b'_1$ .

(1-1) When  $\omega_0$  is a  $p$ -wave along  $b_1$ , then apply the  $p$ -wave move to  $p_0(L)$ . The resulting projection  $p'_0(L)$  will jump to  $p_1(L)$  and  $|p'_0(L)| < |p_1(L)|$ .

$$p_0(L) \xrightarrow{p(\omega_0)} p'_0(L) \xrightarrow{\text{jump}} p_1(L).$$

Suppose that each innermost  $p$ -wave is not along  $b_1$ . Since  $p_1(L)$  must have a  $p$ -wave, the first wave  $\omega_1$  can be taken as  $\omega_0$ .

(1-2) When  $\omega_1$  is a  $p$ -wave along  $b_2$  or  $b_3$ , then it can be regarded as a  $p$ -wave for  $p_0(L)$ . The projection  $p'_0(L)$  obtained from  $p_0(L)$  by the  $p$ -wave move with  $\omega_1$  jumps to  $p_2(L)$ , and  $|p'_0(L)| < |p_0(L)|$ .

$$p_0(L) \xrightarrow{p(\omega_1)} p'_0(L) \xrightarrow{\text{jump}} p_2(L).$$

(1-3) When  $\omega_1$  is a  $p$ -wave along  $b'_1$ , then  $p_0(L)$  jumps directly to  $p_2(L)$  beyond  $p_1(L)$ , and the length  $n$  of the original regular sequence of wave moves decreases by one.

$$p_0(L) \xrightarrow{\text{jump}} p_2(L).$$

In each case, we can find a new sequence of smaller complexity and hence the desired one by the inductive hypothesis. The reader should, however, give attention to the last case (1-3) where our induction depends on the second factor of the complexity.

Hereafter, both of  $p_0(L)$  and  $p_1(L)$  are assumed to be normalized, so the first wave  $\omega_1$  must be an  $s$ -wave by the regularity of the wave move sequence. We shall minimize the number of intersecting points of  $b_1$  and  $\omega_1$  by carrying only  $\omega_1$  isotopically, neglecting  $b_1$  and not deforming  $p_1(L)$ , and make roughly the three cases below.

*Case 2.* The  $s$ -wave  $\omega_1$  intersects  $b_1$  in more than one point. We have two cases given in Figure 4-2. Each subarc of  $\omega_1$ , say  $\omega_0$ , which meets  $b_1$  only in its endpoints is an  $s$ -wave for  $p_0(L)$  since  $p_0(L)$  is normalized and connected. Normalize the result  $p'_0(L)$  of the wave move with  $\omega_0$ ; then a projection equivalent to  $p_1(L)$  will be obtained since  $p_0(L)$  and  $p_1(L)$  are normalized (see Figure 4-3 for the left-hand case in Figure 4-2):

$$p_0(L) \xrightarrow{\omega_0} p'_0(L) \xrightarrow{p} \cdots \xrightarrow{p} p_1(L).$$

We can construct the required wave moves from the above sequence only by joining it to the original one, not using the inductive hypothesis.

Case 3. The  $s$ -wave  $\omega_1$  does not intersect  $b_1$ .

(3-1) When  $\omega_1$  joins an under bridge, then  $\omega_1$  is an  $s$ -wave for  $p_0(L)$ ; otherwise, the subarc  $\beta$  cut off from the under bridge by  $\omega_1$  would not intersect  $b_1$ ,  $b_2$  and  $b_3$ , and a circle along  $\omega_1 \cup \beta$  would separate  $p_0(L)$ , contrary to the connectivity of  $p_0(L)$ . Interchanging the first jump move and the wave move with  $\omega_1$ , we obtain a new sequence

$$p_0(L) \xrightarrow{\omega_1} p'_0(L) \xrightarrow{\text{jump}} p_2(L) \xrightarrow{\omega_2} \cdots \xrightarrow{\omega_n} p_{n+1}(L).$$

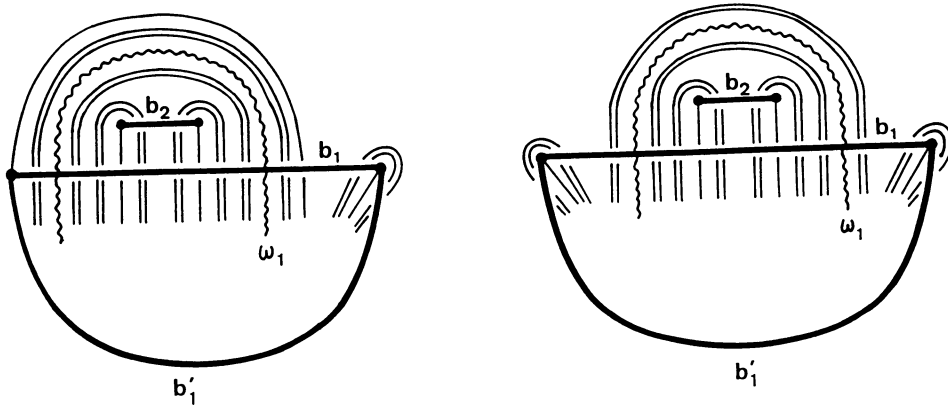


FIGURE 4-2

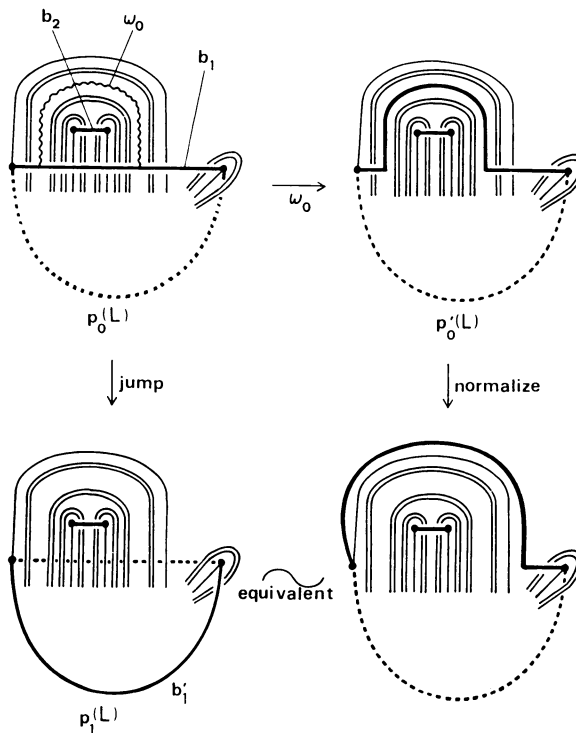


FIGURE 4-3

The inductive hypothesis can be applied to the sequence starting from  $p'_0(L)$ , since  $|p'_0(L)| < |p_0(L)|$ .

(3-2) When  $\omega_1$  joins an over bridge, it must be  $b'_1$  by the normality of  $p_1(L)$ . If there are several under bridges passing through  $b'_1$  parallel to  $\omega_1$ , then  $p_2(L)$  may have a cancelling region adjacent to  $\omega_1$ . By the regularity of the sequence of wave moves and the normality of  $p_{n+1}(L)$ , the consecutive wave moves with  $\omega_2, \dots, \omega_{k-1}$ , for some  $k$ , remove such a cancelling region so that  $p_k(L)$  is equivalent to  $p_0(L)$ . (See Figure 4-4.) The subsequence beginning with  $p_k(L)$  can be thought of as regular wave moves which transform  $p_0(L)$  into  $p_{n+1}(L)$ .

Case 4. The  $s$ -wave  $\omega_1$  intersects  $b_1$  in exactly one point.

(4-1) When  $\omega_1$  joins an over bridge of  $p_1(L)$ , necessarily  $b'_1$ , then  $\omega_1$  has one of two types shown in Figures 4-5 and 4-6. In the former case, it is immediate to see that  $p_0(L)$  is equivalent to  $p_2(L)$  and that the original sequence of regular wave

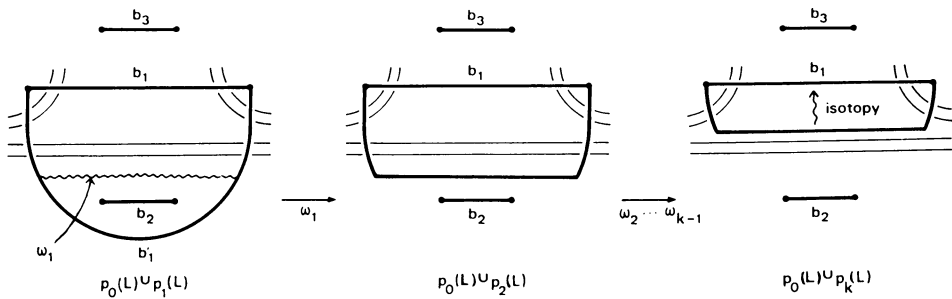


FIGURE 4-4

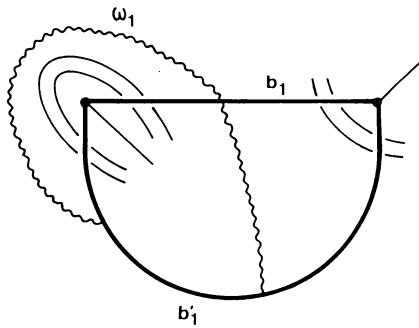


FIGURE 4-5

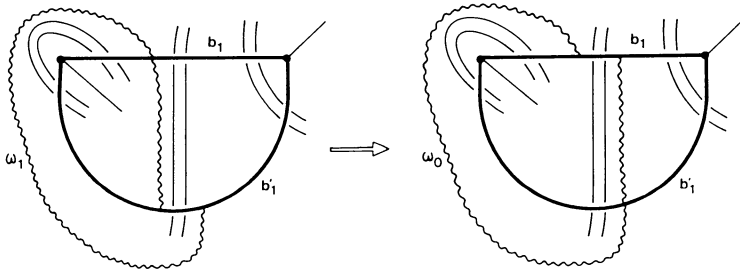


FIGURE 4-6



moves with the first one omitted carries  $p_0(L)$  to  $p_{n+1}(L)$ . In the latter case, if we shrink  $\omega_1$  at one of its ends and extend  $\omega_1$  at the other, then an  $s$ -wave  $\omega_0$  for  $p_0(L)$  joining  $b_1$  will be acquired. (See Figure 4-6.) Since the initial jump move can be regarded as the wave move with  $\omega_0$  up to equivalence, the original regular sequence from  $p_1(L)$  with the  $s$ -wave move  $\omega_0$  added ahead is the required one.

We shall put below the case that an under bridge  $\underline{b}$  of  $p_1(L)$  contains the endpoints of  $\omega_1$ .

(4-2) Suppose that one endpoint of  $\omega_1$  and an interior point of  $b_1$  bound a subarc  $\gamma$  of  $\underline{b}$  whose interior has no crossing of  $p_1(L)$ , as illustrated in Figure 4-7. Slide the endpoint of  $\omega_1$  along  $\gamma$  into the opposite side of  $b_1$ ; then the procedure similar to Case 2 can be used because the number of points in  $\omega_1 \cap b_1$  increases by one.

(4-3) Suppose that one endpoint of  $\omega_1$  and one endpoint of  $b_1$  bound a subarc  $\gamma$  of  $\underline{b}$  whose interior has no crossing of  $p_1(L)$ . It should be distinguished which side of  $\underline{b}$

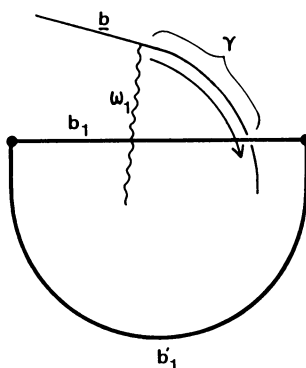


FIGURE 4-7

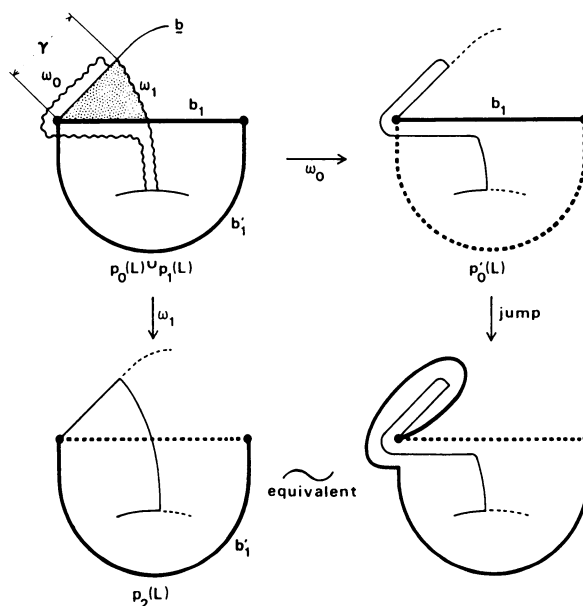


FIGURE 4-8

the  $s$ -wave  $\omega_1$  attaches to. In case of the side facing  $b_1$ , no bridge of  $p_1(L)$  passes through the triangular region surrounded by  $\gamma$ ,  $\omega_1$  and  $b_1$ , because of the minimality of  $\omega_1 \cap b_1$  and the normality of  $p_0(L)$  and  $p_1(L)$ . We find a wave move with an  $s$ -wave  $\omega_0$ , as indicated in Figure 4-8, whose result  $p'_0(L)$ , with  $|p'_0(L)| < |p_0(L)|$ , jumps to a projection equivalent to  $p_2(L)$ :

$$p_0(L) \xrightarrow[\omega_0]{} p'_0(L) \xrightarrow[\text{jump}]{} p_2(L).$$

Use the inductive hypothesis.

In the other case, there arises the situation described in Figure 4-9, where the circle is filled with a neighborhood of an over bridge  $b_2$  or  $b_3$ . (We shall also replace

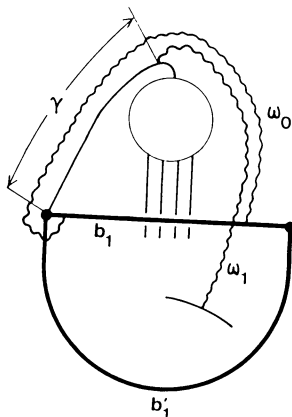


FIGURE 4-9

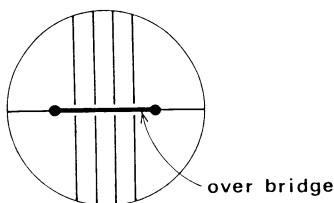


FIGURE 4-10

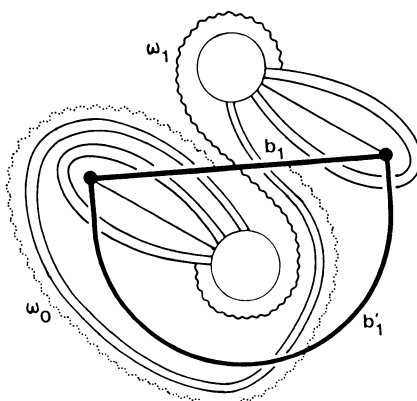


FIGURE 4-11

with the parts, like Figure 4-10, each circle of the succeeding figures in this section.) Then a projection equivalent to  $p_1(L)$  can be obtained from  $p_0(L)$  by a wave move with an  $s$ -wave  $\omega_0$  instead of the jump move, so we have the regular sequence of wave moves

$$p_0(L) \xrightarrow{\omega_0} p_1(L) \xrightarrow{\omega_1} \cdots \xrightarrow{\omega_n} p_{n+1}(L).$$

Now we shall deal with the remaining case, where  $\omega_1$  joins an under bridge  $\underline{b}$  of  $p_1(L)$  in neither of the cases (4-2) and (4-3). Start from each endpoint of  $\omega_1$  and go along  $\underline{b}$  in any direction; then we will encounter a over bridge different from  $b_1$ . Precisely one such over bridge for each starting point must coincide with  $b'_1$ , since  $p_1(L)$  is normalized. Paying attention to the normality of  $p_0(L)$  and  $p_1(L)$ , we conclude that  $p_1(L) \cup b_1 \cup \omega_1$  has one of the forms described in Figures 4-11 to 4-13, where each arc like the letter “S” represents the  $s$ -wave  $\omega_1$ .

(4-4) In the case of Figure 4-11, there is an  $s$ -wave  $\omega_0$  for  $p_0(L)$  whose wave move, instead of the jump move, transforms  $p_0(L)$  into a projection equivalent to  $p_1(L)$ ; observe that at least one arc joins the two circles associated with  $b_2$  and  $b_3$ , crossing  $b_1$  and  $b'_1$ .

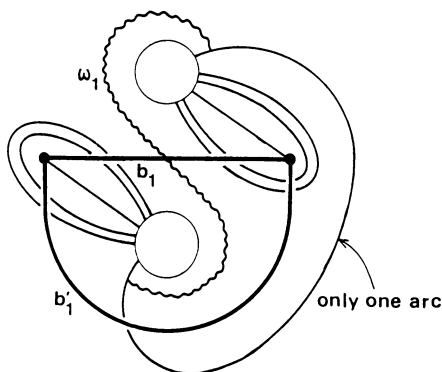


FIGURE 4-12

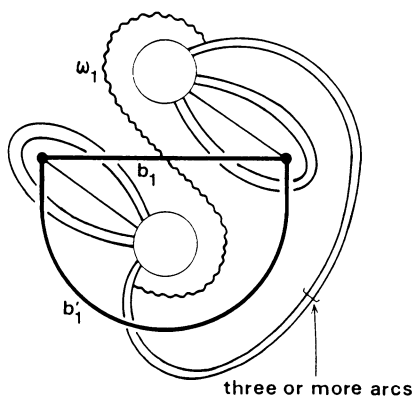


FIGURE 4-13

(4-5) In the case of Figure 4-12, the two circles are connected by only one arc which does not cross  $b_1$ . The replacement of the arc with  $\omega_1$  yields  $p_2(L)$  and an isotopy sends  $p_2(L)$  to  $p_0(L)$ , so they are equivalent.

(4-6) In the case of Figure 4-13, it seems that there would be no wave for  $p_0(L)$  and the previous routines would not be available. Then apply the “fake” wave move with  $\omega_1$  to  $p_0(L)$ ; let  $\underline{b}'$  be the under bridge of  $p_2(L)$  obtained from  $\underline{b}$  by the wave move with  $\omega_1$  and let  $\underline{p}_1(L)$  be the projection obtained from  $p_0(L)$  by replacing  $\underline{b}$  with  $\underline{b}'$ . The transformation of  $p_0(L)$  into  $\underline{p}_1(L)$  (with slight modification) can be considered as an under jump move which does not increase the number of crossings. Since  $\underline{p}_1(L)$  and  $p_2(L)$  have common bridges except  $b_1$  and  $\underline{b}'$ ,  $\underline{p}_1(L)$  jumps to  $p_2(L)$  and hence we have:

$$\begin{array}{ccccccc}
 p_0(L) & \xrightarrow{\text{jump}} & p_1(L) & & & & \\
 \text{jump } \downarrow & & \downarrow \omega_1 & & & & \\
 (\omega_1) & & & & & & \\
 \underline{p}_1(L) & \xrightarrow[\text{jump}]{} & p_2(L) & \xrightarrow[\omega_2]{} & p_3(L) & \xrightarrow[\omega_3]{} \cdots \xrightarrow[\omega_n]{} & p_{n+1}(L)
 \end{array}$$

The bottom horizontal sequence beginning with  $\underline{p}_1(L)$  has smaller complexity than the original one. By the inductive hypothesis, it can be transformed into regular wave moves with  $\underline{\omega}_1, \dots, \underline{\omega}_m$  and we obtain the sequence

$$p_0(L) \xrightarrow[\text{jump}]{} p_1(L) \xrightarrow[\omega_1]{} p_2(L) \xrightarrow[\omega_2]{} \cdots \xrightarrow[\underline{\omega}_m]{} p_{m+1}(L).$$

Without loss of generality,  $\underline{p}_{m+1}(L)$  may be assumed to have the property  $Q$ , because the disconnectivity is  $W$ -admissible. Notice that if the length  $m$  of the new sequence is less than  $n$  then the inductive hypothesis can be immediately used, but  $m$  may exceed  $n$  possibly.

Now return to the beginning of this proof and repeat the same argument for the new sequence with the role of over and under bridges interchanged. Recall that in each case, except (1-3) and (4-6), we obtained the desired sequence, not depending on the second factor of the complexity ( $|p_0(L)|, n$ ). We shall observe that neither (1-3) nor (4-6) happens for the new sequence.

Figures 4-14 and 4-15 suggest the relationship of  $p_0(L)$  and  $\underline{p}_1(L)$ . The left- or right-hand in each figure shows it according as  $\underline{p}_1(L)$  is regarded as the result of the “fake” wave move with  $\omega_1$  for  $p_0(L)$  or not. Recall that  $\omega_1$  meets  $b_1$  in exactly one

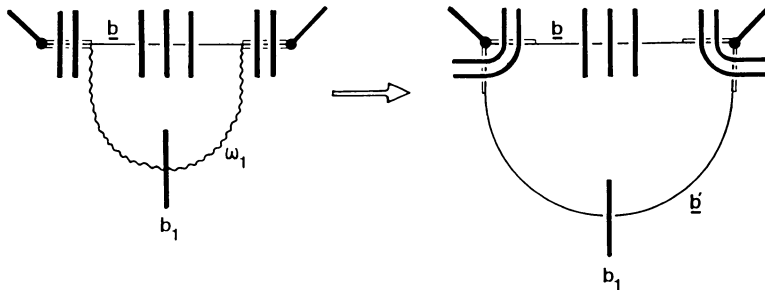


FIGURE 4-14

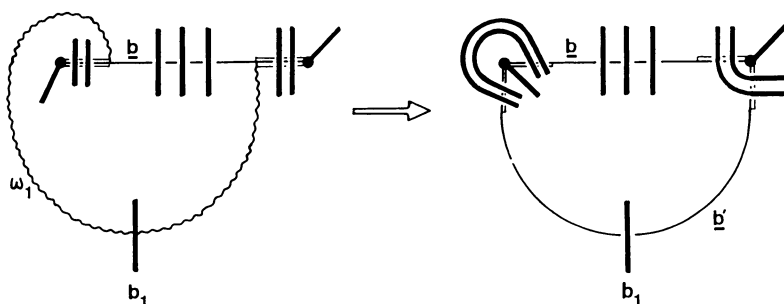


FIGURE 4-15

point in Case 4. It follows that the new sequence must not admit the case (4-6); if it did, then under bridges inside and outside  $\underline{b} \cup \underline{b}'$  would have to be joined by three or more subarcs or over bridges crossing  $\underline{b}'$ , but only one arc indicated by  $b_1$  in each figure is such a subarc.

Suppose that the case (1-3) held for the new sequence. Then  $\omega_1$  would be a  $p$ -wave along  $\underline{b}'$  joining  $b_1$  by the normality of  $p_0(L)$ , and  $\underline{b}'$  would surround the cancelling region together with a subarc of  $b_1$ . One of the endpoints of the subarc is the single point in  $\omega_1 \cap b_1$  and the other is contained in the bold part of  $\underline{b}'$ . This implies that  $\omega_1$  would be an  $s$ -wave for  $p_1(L)$  treated in (4-2) and (4-3), where we are not standing but in (4-6) for the original sequence. So (1-3) is not the case for the new sequence.

Therefore we can transform the new sequence into the desired one, using only the inductive hypothesis on the first factor of the complexity. Now our induction is completed. ■

**5. Applications to Heegaard diagrams.** This section plays a role to translate our results on 3-bridge links into those on orientable closed 3-manifolds with genus 2 Heegaard splittings and presents our proofs of Theorems 1-3 and 1-4.

Let  $M^3$  be an orientable closed 3-manifold and  $F^2$  an orientable closed surface in  $M^3$ . The pair  $(M^3, F^2)$  is called a *Heegaard splitting* for  $M^3$  with *splitting surface*  $F^2$  if  $F^2$  splits  $M^3$  into two handlebodies  $U$  and  $V$ . The *genus* of the splitting  $(M^3, F^2)$  is defined as the number of handles of  $U$  (or  $V$ ), which is called also the *genus* of  $U$ . The pair  $(M^3, F^2)$  is assumed below to be a Heegaard splitting of genus 2.

A proper disk in a handlebody which cuts its handle is called its *meridian disk*. A genus 2 handlebody has two disjoint meridian disks which cut it into a 3-ball. Let  $u_1, u_2$  and  $v_1, v_2$  be the boundaries of such meridian disks of  $U$  and  $V$ , respectively, which lie on the splitting surface  $F^2$ . We call  $H = (u_1, u_2; v_1, v_2)$  a (genus 2) *Heegaard diagram* of the splitting  $(M^3, F^2)$  and call  $u_1, u_2$  *meridians* and  $v_1, v_2$  *longitudes* of  $H$ .

There is another circle  $u_3$  on  $F^2$ , disjoint from and nonparallel to  $u_1$  and  $u_2$ , which bounds a meridian disk of  $U$ , and there is similar circle  $v_3$  on  $F^2$  for  $V$ . Then  $\tilde{H} = (u_1, u_2, u_3; v_1, v_2, v_3)$  is called an *extended Heegaard diagram* of  $(M^3, F^2)$  or an *extension* of  $H$  with the notation  $\tilde{H} = H(u_3, v_3)$ . Note that nine Heegaard diagrams admit  $\tilde{H}$  as their extensions.

Two Heegaard splittings are *equivalent* if there is a homeomorphism between 3-manifolds which sends one of the splitting surfaces onto the other. The following fact in [BH, Theorem 8] connects 3-bridge decompositions of links to genus 2 Heegaard splittings of orientable closed 3-manifolds:

**PROPOSITION 5-1 (BIRMAN AND HILDEN).** *There exists a one-to-one correspondence between equivalence classes of genus 2 Heegaard splittings and those of 3-bridge links with decomposing spheres.*

Let  $L$  be a 3-bridge link in  $S^3$  with decomposing sphere  $S^2(L)$  and  $q: M^3 \rightarrow S^3$  be the 2-fold branched covering of  $S^3$  branched over  $L$ . Then  $F^2 = q^{-1}(S^2(L))$  is a

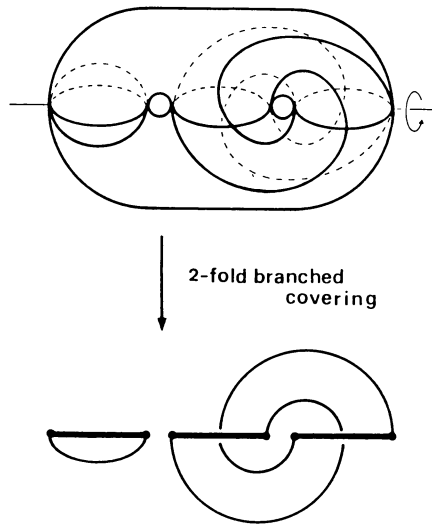


FIGURE 5-1

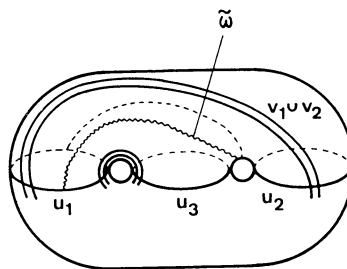


FIGURE 5-2

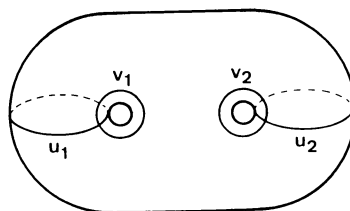


FIGURE 5-3

genus 2 Heegaard surface of  $M^3$ . The 3-bridge decomposition  $(L, S^2(L))$  and the genus 2 Heegaard splitting  $(M^2, F^2)$  are related by the correspondence in Proposition 5-1. Furthermore, there is a correspondence between 3-bridge projections and extended Heegaard diagrams of genus 2, as follows. Let  $p(L)$  be a 3-bridge projection of  $L$  on  $S^2(L)$  with over bridges  $b_1^+, b_2^+, b_3^+$  and under bridges  $b_1^-, b_2^-, b_3^-$ . It is easy to see that each  $u_i^\pm = q^{-1}(b_i^\pm)$  is a circle on  $F^2$  and that  $\tilde{H} = (u_1^+, u_2^+, u_3^+; u_1^-, u_2^-, u_3^-)$  is an extended Heegaard diagram of  $(M^3, F^2)$ . Then  $\tilde{H}$  is said to be *associated with*  $p(L)$ . For example, Figure 5-1 illustrates the relationship of a disconnected 3-bridge projection and the extended Heegaard diagram for  $S^2 \times S^1 \# L(2, 1)$  associated with it.

The *complexity*  $c(H)$  of a genus 2 Heegaard diagram  $H = (u_1, u_2; v_1, v_2)$  is defined as the cardinality of  $(u_1 \cup u_2) \cap (v_1 \cup v_2)$ . Similarly define the *complexity*  $c(\tilde{H})$  of an extended diagram  $\tilde{H}$ . Let  $\omega$  be an arc on  $F^2$  such that for a meridian or a longitude of  $H$ , say  $u_1$ ,  $\omega \cap (u_1 \cup u_2 \cup v_1 \cup v_2) = \omega \cap u_1 = \partial\omega$  and both ends of  $\omega$  attach to the same side of  $u_1$ . Then one of two circles in  $u_1 \cup \omega$  different from  $u_1$  bounds a meridian disk of  $U$ , say  $u_3$ , and  $H' = (u_3, u_2; v_1, v_2)$  is a new Heegaard diagram of  $(M^3, F^2)$ . We call  $\omega$  a *wave* for  $H$  and the replacement of  $u_1$  with  $u_3$  a *wave move* with  $\omega$  if  $c(H') < c(H)$ .

A wave move simplifies a Heegaard diagram as well as a wave move for a 3-bridge projection of a link. There is not a one-to-one correspondence between them, but they are related as follows:

**PROPOSITION 5-2.** *Let  $(M^3, F^2)$  be a genus 2 Heegaard splitting for  $M^3$  and  $L$  a 3-bridge link associated with  $(M^3, F^2)$ . Let  $Q$  be a  $W$ -admissible property. Suppose that there is at least one 3-bridge projection of  $L$  on its decomposing sphere which has the property  $Q$ . Then one of the following three conditions holds for any Heegaard diagram  $H$  of  $(M^3, F^2)$ :*

- (i)  $H$  has a wave.
- (ii) There is a meridian (or a longitude) of  $H$  which intersects no longitude (or no meridian).
- (iii) There is an extension  $\tilde{H}$  of  $H$  which is associated with a 3-bridge projection of  $L$  having the property  $Q$ .

Note that the condition (ii) implies  $M^3$  having  $S^2 \times S^1$  as a factor of its connected sum decomposition.

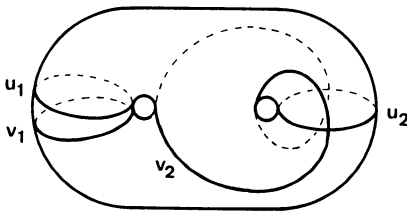


FIGURE 5-4

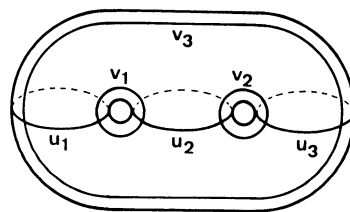


FIGURE 5-5

PROOF. Let  $H = (u_1, u_2; v_1, v_2)$  be a Heegaard diagram for  $(M^3, F^2)$  which satisfies none of the first two conditions (i) and (ii). Choose an extension  $\tilde{H} = H(u_3, v_3)$  of  $H$  with  $u_3 \cap (v_1 \cup v_2 \cup v_3) \neq \emptyset$  and  $v_3 \cap (u_1 \cup u_2 \cup u_3) \neq \emptyset$ , so as to minimize  $c(\tilde{H})$ . If a 3-bridge projection  $p(L)$  of  $L$  associated with  $\tilde{H}$  does not have the property  $Q$  then  $p(L)$  has a wave  $\omega$  by Proposition 4-2. The wave  $\omega$  joins one of the bridges associated with  $u_1, u_2, v_1$  and  $v_2$ , say  $u_1$ , since  $c(\tilde{H})$  is minimum.

Let  $\tilde{\omega}$  be one of lifts of  $\omega$  on  $F^2$ . Since  $H$  has no wave and  $u_2 \cap (v_1 \cup v_2) \neq \emptyset$ , there are several subarcs of  $v_1$  and  $v_2$  joining  $u_1$  and  $u_2$  as shown in Figure 5-2. This implies that  $\tilde{\omega}$  would be a wave for  $H$  which transforms  $u_1$  into  $u_3$ , because of the minimality of  $c(\tilde{H})$ . It is a contradiction. Therefore  $p(L)$  has the property  $Q$ , so (iii) holds for  $H$ . ■

In the remaining part of this section, we shall prove Theorems 1-3 and 1-4, using Proposition 5-2. First, we should define the standard forms of genus 2 Heegaard diagrams for  $S^3$  and  $S^2 \times S^1 \# L(p, q)$ .

The *standard form* for  $S^3$  is a unique diagram  $H_0 = (u_1, u_2; v_1, v_2)$  shown in Figure 5-3, where  $u_i \cap v_j = \emptyset$  ( $i \neq j$ ) and  $u_i \cap v_i$  consists of only one point. A Heegaard diagram  $H = (u_1, u_2; v_1, v_2)$  for  $S^2 \times S^1 \# L(p, q)$  is *standard* if  $u_1$  and  $v_1$  are parallel to each other and are disjoint from  $u_2$  and  $v_2$  and if  $u_2 \cap v_2$  consists of exactly  $p$  points. The standard forms for  $S^2 \times S^1 \# L(p, q)$  are not unique if  $p \neq 1$ . Note that a canonical genus 1 Heegaard diagram  $(u_2; v_2)$  for  $L(p, q)$  is obtained from each standard Heegaard diagram for  $S^2 \times S^1 \# L(p, q)$  by cutting a handle of its splitting surface along  $u_1$  (or  $v_1$ ). For example, the Heegaard diagram in Figure 5-4 is a standard one for  $S^2 \times S^1 \# L(2, 1)$ .

As is mentioned in introduction,  $S^3$  and  $S^2 \times S^1 \# L(p, q)$  can be represented as the 2-fold branched covering of  $S^3$  branched over a trivial knot and a splittable 3-bridge link. Conversely, we shall observe the uniqueness of such branch lines. For  $S^3$ , it follows from Waldhausen's result about involutions on  $S^3$  [W]:

PROPOSITION 5-3 (WALDHAUSEN). *The 2-fold branched covering of  $S^3$  branched over a link is homeomorphic to  $S^3$  if and only if the link is a trivial knot.*

In order to see the uniqueness for  $S^2 \times S^1 \# L(p, q)$ , we shall use the following result on lens spaces, due to Hodgson [Ho] in the spherical cases  $p \neq 0$  and due to Tollefson [T] for  $S^2 \times S^1$ .

PROPOSITION 5-4 (HODGSON AND TOLLEFSON). *The 2-fold branched covering of  $S^3$  branched over a link is homeomorphic to a lens space  $L(p, q)$  if and only if the link is equivalent to a 2-bridge link  $K(p, q)$ .*

PROPOSITION 5-5. *The 2-fold branched covering of  $S^3$  branched over a link is homeomorphic to  $S^2 \times S^1 \# L(p, q)$  if and only if the link is equivalent to a splittable union of a trivial knot and a 2-bridge link  $K(p, q)$ .*

PROOF. Let  $M^3(L)$  denote the 2-fold branched covering of  $S^3$  branched over a link  $L$ . We have the following two formulas:

- (i)  $M^3(L) = M^3(L_1) \# M^3(L_2) \# S^2 \times S^1$  if  $L$  is a splittable union  $L_1 \cup L_2$ .
- (ii)  $M^3(L) = M^3(L_1) \# M^3(L_2)$  if  $L$  is a connected sum  $L_1 \# L_2$ .



Sufficiency is an immediate consequence of the first formula; take a trivial knot and  $K(p, q)$  as  $L_1$  and  $L_2$ , respectively, and use Propositions 5-3 and 5-4.

In order to show necessity, we shall apply the  $\mathbf{Z}_2$ -equivariant sphere theorem [KT] to the covering translation  $\tau$  on  $M^3(L) = S^2 \times S^1 \# L(p, q)$ . Since  $M^3(L)$  is not irreducible, there is an  $\tau$ -equivariant 2-sphere  $S^2$  in  $M^3(L)$  which bounds no 3-ball and the projection of  $S^2$  decomposes  $L$  into either  $L_1 \cup L_2$  or  $L_1 \# L_2$ . By the uniqueness of prime decompositions [Mil],  $M^3(L_1) = S^3$  and  $M^3(L_2) = L(p, q)$  in the formula (i) and  $M^3(L_1) = S^2 \times S^1$  and  $M^3(L_2) = L(p, q)$  in the formula (ii). In either case, we shall conclude from Propositions 5-3 and 5-4 that  $L$  is a splittable union of a trivial knot and  $K(p, q)$ . ■

Now all we need have been prepared and the proofs of Theorems 1-3 and 1-4 will be completed.

**PROOF OF THEOREM 1-3.** Take  $S^3$  as  $M^3$  in Proposition 5-2, then  $L$  is equivalent to a 3-bridge trivial knot by Proposition 5-3. Take the triviality of projections as  $\mathcal{Q}$ . Proposition 3-1 guarantees the hypothesis of  $L$  in Proposition 5-2. So any genus 2 Heegaard diagram for  $S^3$  satisfies (i) or (iii). If (iii) holds then  $\tilde{H}$  is associated with a trivial 3-bridge projection and is equivalent to the diagram in Figure 5-5. Each diagram obtained from  $\tilde{H}$  by deleting any  $u_i$  and  $v_j$  is equivalent to the standard Heegaard diagram for  $S^3$  or has a wave, and so  $H$  is or does. ■

**PROOF OF THEOREM 1-4.** Take  $S^2 \times S^1 \# L(p, q)$  as  $M^3$  in Proposition 5-2, then  $L$  is equivalent to a splittable union of a trivial knot and a 2-bridge link  $K(p, q)$  by Proposition 5-5. Take the disconnectivity of projections as  $\mathcal{Q}$ . The hypothesis of  $L$  in Proposition 5-2 follows from Proposition 3-2. Let  $H = (u_1, u_2; v_1, v_2)$  be a genus 2 Heegaard diagram for  $S^2 \times S^1 \# L(p, q)$  with splitting surface  $F^2$ . It suffices to find a wave for  $H$  in only two cases (ii) and (iii) of Proposition 5-2, assuming that there is no parallel pair of a meridian and a longitude in  $H$ .

Suppose that  $u_1$  is a meridian of  $H$  which intersects no longitudes. Then there is an arc  $\alpha$  on  $F^2$  which joins one of  $v_1$  and  $v_2$ , say  $v_1$  to  $u_1$  and whose interior does not intersect  $u_1 \cup u_2 \cup v_1 \cup v_2$  and there is a wave for  $H$  which starts from  $v_1$ , goes along near  $\alpha$ , turns around  $u_1$  and returns back to  $v_1$  along  $\alpha$ . The wave move transforms  $v_1$  into a loop parallel to  $u_1$ , so we found a wave for  $H$  in case of (ii).

Now put the case (iii), excluding (ii). Then the extension  $\tilde{H}$  of  $H$  is associated with a standard disconnected 3-bridge projection  $p(L)$  of a splittable link, that is, a disjoint union of a 1-bridge circular projection and Schubert's 2-bridge form. Let  $u_*$  and  $v_*$  denote the two loops in  $F^2$  as the lifts of the 1-bridge trivial component of  $p(L)$ . The loops  $u_*$  and  $v_*$  intersect transversely in exactly two points. Clearly  $u_i$  and  $v_j$  are transformed into a parallel pair by a wave move when  $u_i = u_*$  and  $v_j = v_*$  ( $i, j = 1, 2$ ). Otherwise, there is an arc which joins one of  $v_1$  and  $v_2$  to  $v_*$  and we can find a wave for  $H$  similarly to the case (ii). ■

**6. Observations and examples.** In this section, we shall observe the possibility to extend our results, showing several examples.

Do Theorems 1-1 and 1-2 hold for projections of links with bridge number more than 3? There has been shown, by Morikawa [Mor], a 4-bridge nontrivial projection of a trivial knot which has no wave. Figure 6-1 presents three 4-bridge projections of

a splittable link  $K(3,1) \cup K(3,1)$ . The left-hand has no wave and jumps to the middle, which is transformed into the right-hand disconnected one by a wave move. These are counterexamples to an extension of Lemma 4-3 rather than that of Theorem 1-2. So the answer to the question is “NO”.

Recall that Theorems 1-1 and 1-2 are only immediate consequences of Proposition 4-2. So we have the second question: What other property is  $W$ -admissible?

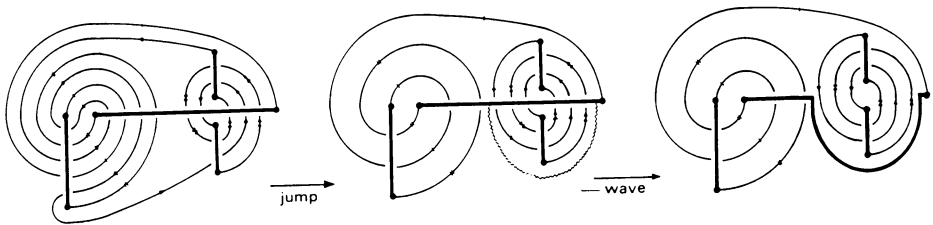


FIGURE 6-1

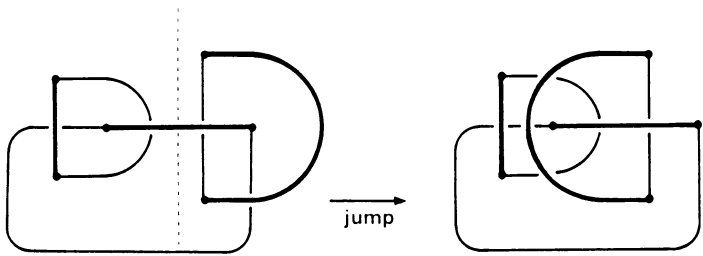


FIGURE 6-2

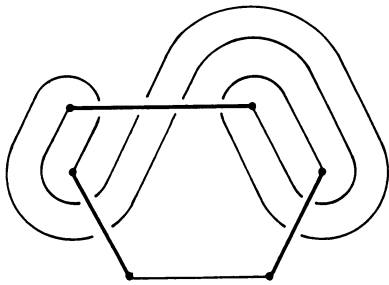


FIGURE 6-3

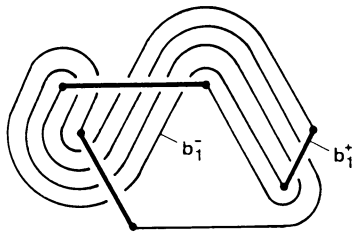


FIGURE 6-4

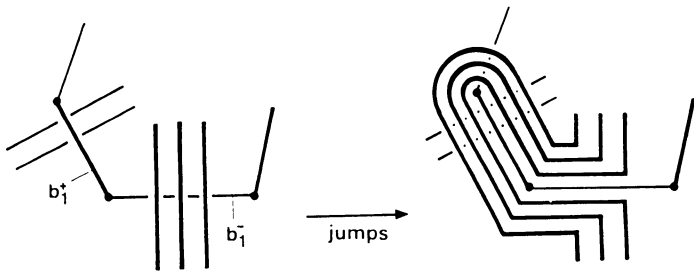


FIGURE 6-5

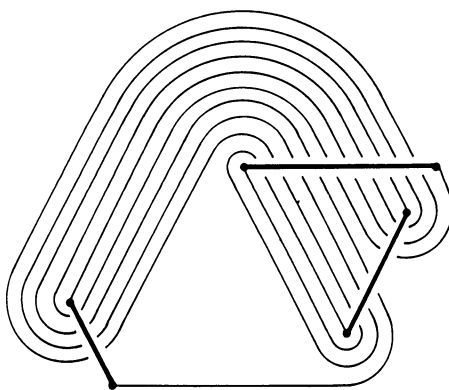


FIGURE 6-6

It might be expected that the primarity of a 3-bridge link could be recognized by wave moves. Unfortunately, Figure 6-2 gives a counter example to the expectation. The left-hand projection describes just a connected sum decomposition of the composite link  $K(2, 1) \# K(2, -1)$  and jumps to the right-hand one which has no wave. Therefore, having a form to describe a connected sum decomposition of a link, like the left-hand of Figure 6-2, is not a  $W$ -admissible property.

Are wave moves available for deciding whether a given 3-bridge link admits a 2-bridge decomposition? The 3-bridge projection in Figure 6-3 is transformed into Schubert's form of the 2-bridge knot  $K(5, 4)$  by eliminating an under bridge, and jumps to the 3-bridge projection in Figure 6-4 which has no wave. Besides, none of its bridges can be eliminated in the obvious way. There is, however, a way to reduce it into a 2-bridge projection, as follows.

First, find a pair of an over and under bridge which have only one common endpoint and no other intersection, say  $b_1^+$  and  $b_1^-$ . We call such a pair a *cancelling pair*. Next, jump all over bridges crossing  $b_1^-$  beyond  $b_1^+$  in order, as in Figure 6-5, then we can eliminate  $b_1^-$ . In fact, the projection in Figure 6-4 has a cancelling pair,  $b_1^+$  and  $b_1^-$ .

Can a 3-bridge projection for a 2-bridge link be always transformed into one which has a cancelling pair by wave moves? Figure 6-6 gives a negative example where there is no wave and no cancelling pair. The projection represents also  $K(5, 4)$ . Therefore, the existence of a cancelling pair is not a  $W$ -admissible property, too.

Recently, Negami [N] has proved that every 3-bridge projection of the Hopf link can be transformed into a unique projection with precisely two crossings by a finite sequence of wave moves, and has observed that there are few  $W$ -admissible properties in a sense. Notice that the above observations can be translated into those on Heegaard diagrams by Proposition 5-2. So a new strategy different from a wave move will be needed for other decision problems of links and also of 3-manifolds.

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